Steady-state density perturbations induced by a point mass in a finite cylinder

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Abstract

We consider a massive point particle moving along a circular orbit, immersed in a finite cylindrical gaseous environment, as a model for astrophysical compact objects embedded in a gaseous disk. We compute, via a frequency-domain calculation, the density perturbation induced by the gravitational influence of the perturber as a first-approximation to the full hydrodynamical problem.

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1 Introduction

The interaction of a massive moving object (hereafter referred to as *perturber*) immersed in a gaseous medium with its gravitationally induced wake gives rise to a momentum loss commonly known as dynamical friction (hereafter referred to as DF). DF was first studied by Subrahmanyan Chandrasekhar in 1943 for the collisionless background case [1]. This led to an investigation of DF for straight-line motion in gaseous backgrounds by Ostriker [2], which was eventually extended to the circular orbit case by Kim & Kim [3]. Its consequences have been significant in modelling the formation and evolution of planets and stars [4, 5], and the evolution of galactic mergers [6], among many others. However, the main assumption in the usual treatment is that the characteristic length of the medium is much larger than that of the orbital radius of the moving object; that is, the background medium is infinite. In the context of astrophysical bodies residing in finite gaseous disks, this formulation of DF may not completely describe the dynamics of the system. Hence, there is a need to extend the infinite background case to when the characteristic length of the medium is comparable to that of the orbital radius of an orbiting body.

Vicente, Cardoso and Zilho (hereafter VCZ) derived expressions for DF in finite slab geometries which addresses the need for a more realistic model than that of an infinite ambient medium [7]. The study of DF in compact geometries was already done before by Namouni [8]; however, only one wake reflection was considered using Neumann boundary conditions. VCZ computed for the DF by imposing Dirichlet and Neumann boundary conditions at the sharp boundaries of the slab geometry and found out that there are episodes in which the drag experienced by a perturber moving on a straight line is along the direction of its trajectory; that is, the DF effect accelerates the perturber. This can be attributed to regions in which the fluid perturbation is negative (i.e. underdensities) due to perturbations reflected from the boundaries. VCZ pointed out that these wake reflections tend to suppress the effect of DF.

In this work, we propose a *purely analytic* method to obtain a first approximation to the gas density profiles induced by a massive point particle inside a finite cylinder. This serves as a first step in calculating for the DF experienced by an astrophysical object embedded in a gaseous disk. In the following calculation, we consider a finite pressure-release cylindrical background of length L and radius a.

2 Solving for the density perturbation function

It was shown in [2] that the linearized fluid equations can be recasted into a wave equation sourced by a gravitational potential,

$$\left(\nabla^2 - \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}\right)\psi(r,\theta,z,t) = -\frac{4\pi G}{c_s^2}\rho_{\text{pert}} \tag{1}$$

where ψ is the density perturbation function, c_s is the speed of sound in the fluid, and ρ_{pert} is the density of a massive point particle moving along a circular orbit with radius R_o, axial position z_o, and angular frequency Ω, which takes the form

$$\rho_{\text{pert}} = \frac{M}{r}\delta(r-R_o)\delta(z-z_o)\delta(\theta-\Omega t) \tag{2}$$

We can decompose the Dirac delta functions in terms of a complete basis while satisfying the boundary condition that we wish to impose. In this problem, we argue that the physically relevant boundary condition is of Dirichlet (reflecting) type, in which the solutions identically vanishes at the boundary. Doing so yields an expansion of the form,

$$\rho_{\text{pert}}(r,\theta,z,t) = \frac{M}{\pi R_o L}\delta(r-R_o)\left(\sum_{m=0}^{\infty}\epsilon_m \cos[m(\theta-\Omega t)]\right)\left(\sum_{n=1}^{\infty}\sin\left[\frac{n\pi}{2L}(z+L)\right]\sin\left[\frac{n\pi}{2L}(z_o+L)\right]\right) \quad (3)$$

where L and R_o are the length of the cylinder and orbital radius, respectively, and

$$\epsilon_m = \begin{cases} 1, & m=0 \\ 2 & m\neq 0 \end{cases} \quad (4)$$

By inspection, we can write down the solution ψ as a summation which is similar in form to that of the source,

$$\psi(r,\theta,z,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\epsilon_m R_{mn}(r)\cos[m(\theta-\Omega t)]\sin\left[\frac{n\pi}{2L}(z+L)\right] \quad (5)$$

Substituting this back to the wave equation then yields a radial equation for $R_{mn}(r)$,

$$\frac{d^2R_{mn}}{dr^2} + \frac{1}{r}\frac{dR_{mn}}{dr} + \left(\alpha_{mn}^2 - \frac{m^2}{r^2}\right)R_{mn} = \frac{-4GM}{c_s^2 R_o L}\delta(r-R_o)\sin\left[\frac{n\pi}{2L}(z_o+L)\right] \quad (6)$$

the general solution of which is a superposition of the Bessel functions of the first and second kind,

$$R_{mn}(r) = A_{mn}J_m(\alpha_{mn}r) + B_{mn}Y_m(\alpha_{mn}r) \quad (7)$$

where

$$\begin{aligned}\alpha_{mn} &= \sqrt{\left(\frac{m\Omega}{c_s}\right)^2 - \left(\frac{n\pi}{2L}\right)^2} \\ &= \frac{1}{R_o}\sqrt{(m\mathcal{M})^2 - \left(\frac{n\pi R_o}{2L}\right)^2}\end{aligned} \quad (8)$$

in which $\mathcal{M}$ is the Mach number, and A_{mn} & B_{mn} are obtained by imposing regularization and boundary conditions. Notice that α_{mn} yields real values only when,

$$\frac{m}{n} > \frac{1}{\mathcal{M}}\frac{\pi R_o}{2L} \quad (9)$$

Hence, we must rewrite equation (6) in the form of the modified Bessel equation in order to ensure that the solutions that we obtain are manifestly real. Doing so yields a solution of the form,

$$R_{mn}(r) = C_{mn}I_m(\beta_{mn}r) + D_{mn}K_m(\beta_{mn}r) \quad (10)$$

where

$$\beta_{mn} = \frac{1}{R_o}\sqrt{\left(\frac{n\pi R_o}{2L}\right)^2 - (m\mathcal{M})^2} \quad (11)$$

for cases when,

$$\frac{m}{n} < \frac{1}{\mathcal{M}}\frac{\pi R_o}{2L} \quad (12)$$

Because of the delta source on the right hand side of (6), the solution, despite being continuous inside the bounded domain, has a discontinuity in its derivatives at $r = R_o$. The jump discontinuity in the derivatives is given by,

$$\left.\left(\frac{dR^{>}_{mn,\alpha/\beta}}{dr} - \frac{dR^{<}_{mn,\alpha/\beta}}{dr}\right)\right|_{r=R_o} = \frac{-2GM}{c_s^2 R_o L}\sin\left[\frac{n\pi}{2L}(z_o+L)\right] \quad (13)$$

where we write,

$$R^{<}_{mn,\alpha}(r) = A_{mn}J_m(\alpha_{mn}r) + B_{mn}Y_m(\alpha_{mn}r) \tag{14}$$
$$R^{>}_{mn,\alpha}(r) = C_{mn}J_m(\alpha_{mn}r) + D_{mn}Y_m(\alpha_{mn}r) \tag{15}$$

and

$$R^{<}_{mn,\beta}(r) = E_{mn}I_m(\beta_{mn}r) + F_{mn}K_m(\beta_{mn}r) \tag{16}$$
$$R^{>}_{mn,\beta}(r) = G_{mn}I_m(\beta_{mn}r) + H_{mn}K_m(\beta_{mn}r) \tag{17}$$

where J_m and Y_m are Bessel function of the first and second kind; and, I_m and K_m are modified Bessel functions of the first and second kind.

In order to ensure that the solution is regular at $r = 0$, we set $B_{mn} = F_{mn} = 0$ since $Y_m(\alpha_{mn}r)$ and $K_m(\beta_{mn}r)$ diverges at the origin. The rest of the coefficients can be obtained by imposing the continuity conditions and Dirichlet boundary condition at $r = a$.

3 Results

Plots of the cross-section along $(\theta, z) = (0, 0)$ over different values of r, as shown in Figure 1, confirms that we have successfully matched the solutions inside and outside the orbital radius R_o.

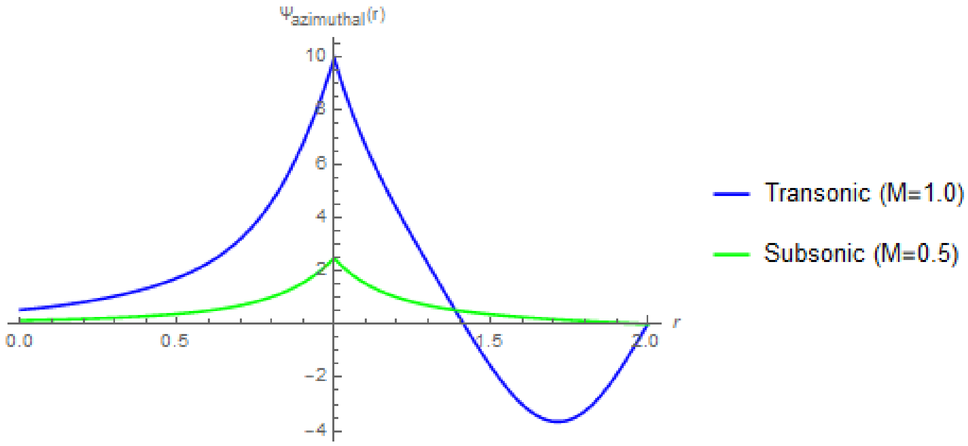


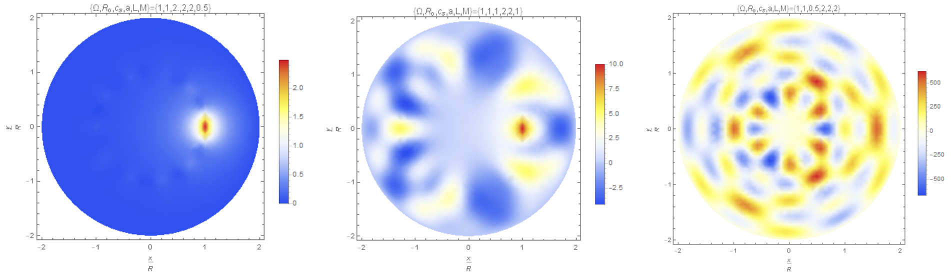
Figure 1: Plot of the density cross-section over r along (θ, z)=$(0, 0)$ for subsonic and transonic perturbers.

In order to compare what we have obtained with the infinite domain result, density plots from our previous work [9] are presented alongside the bounded profiles. We show subsonic ($\mathcal{M} < 0.8$) and transonic ($\mathcal{M} = 0.8 - 1.1$) cases in Figure 2. Supersonic cases were omitted from the analysis because the series solution appear to diverge in this speed regime. Notice that the density trail feature in the infinite medium case cannot be observed in the bounded case. This may be due to the presence of the reflecting boundaries at the walls and endcaps of the cylindrical background. Also, unlike in the infinite medium case, the region surrounding the perturber has underdensities (negative density perturbation) which may act to suppress the effect of overdensities which effectively decelerates the motion of the massive body.

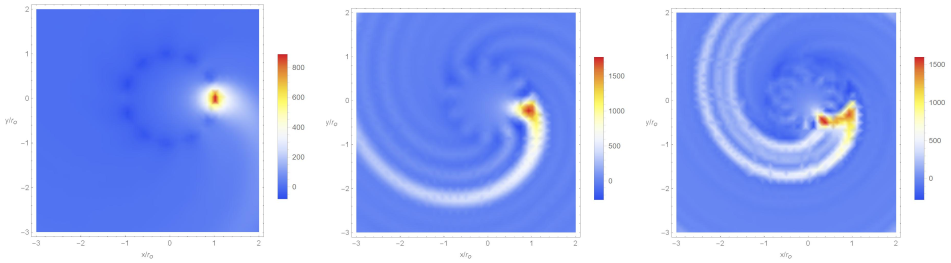
We note, however, that since we used Bessel functions in the construction of our solution, we cannot readily extend this formulation to the infinite case. One could consider using Hankel functions in the expansion which are also solutions of the Bessel differential equation.

4 Conclusion

We have obtained an expression for the steady-state density perturbation induced by a massive point particle moving along a circular orbit immersed in a finite cylindrical gaseous background. This analysis, however, is limited to subsonic to transonic regimes only. To address the divergence in the supersonic regime, a different expansion can be used for the series solution to the wave equation. One striking difference between the density profiles in the infinite and bounded cases is the absence of a density trail, which is present in the infinite medium case. Also, there are regions in which the density perturbation is negative in the bounded medium case. This implies that the presence of reflecting boundaries may suppress the dynamical drag experienced by a massive body along its trajectory. The density perturbation solution obtained in this calculation can be used to compute the force felt by a perturber due to the formation of over(under)-densities around the perturber.



(a) Bounded subsonic perturber (b) Bounded transonic perturber (c) Bounded supersonic perturber



(d) Free subsonic perturber (e) Free transonic pertuber (f) Free supersonic perturber

Figure 2: Plots of the steady-state density profiles induced by a bounded (a) subsonic ($\mathcal{M} = 0.5$), (b) transonic ($\mathcal{M} = 1.0$), (c) supersonic ($\mathcal{M} = 2.0$) perturber alongside density profiles for a free (d) subsonic, (e) transonic, and (f) supersonic perturber. Density trails present in the free perturber case cannot be observed in the bounded case. Also, regions with negative perturbations (underdensities) exist in the vicinity of the massive perturber. For Figures 2(a)-(c), the perturber has an orbital radius $R_o = 1.0$, and cylinder radius and length $a = L = 2.0$.

Acknowledgments

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